

Non-standard scaling law of fluctuations in finite-size systems of globally coupled oscillators

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Universal scaling laws form one of the central issues in physics. A non-standard scaling law or a breakdown of a standard scaling law, on the other hand, can often lead to the finding of a new universality class in physical systems. Recently, we found that a statistical quantity related to fluctuations follows a non-standard scaling law with respect to system size in a synchronized state of globally coupled non-identical phase oscillators [Nishikawa et al., *Chaos* **22**, 013133 (2012)]. However, it is still unclear how widely this non-standard scaling law is observed. In the present paper, we discuss the conditions required for the unusual scaling law in globally coupled oscillator systems, and we validate the conditions by numerical simulations of several different models.

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I. INTRODUCTION

One of the essential studies of physics is to reveal universal phenomena in nature [1]. Synchronization is a universal concept in nonlinear science that is widely used to explain coherent behavior in a system of interacting elements [2] such as a reaction–diffusion system of chemical oscillators [3], an array of electrical units [4], spontaneous luminescence of fireflies [5], and a network of rhythmic biological components [6]. A synchronization transition in a system of N coupled oscillators takes place when the coupling strength surpasses a critical value at which an order parameter changes from zero to a non-zero value. Universality of synchronization phenomena can be partially confirmed by the fact that the scaling property of the order parameter in the vicinity of the critical synchronization transition point does not depend on the details of the system elements [2, 7–20]. In the thermodynamic limit $N \rightarrow \infty$, the study of the universal scaling law with respect to the coupling strength has been a topic of primary interest [7, 8]. On the other hand, many studies have also focused on the universal scaling law with respect to finite system size N . In fact, it is known that the variance of the order parameter generally decays with N in the order of $1/N$ independently of the details of the system elements [9–12]. Such a scaling law is regarded as standard because it applies to a wide range of models. In the present paper, we focus on a non-standard scaling law that may provide an opportunity to determine a new universality class.

To illustrate a difference between standard and non-standard scaling laws, we consider the globally coupled phase oscillator model [14], described as follows:

$$\dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{k=1}^N h(\theta_k - \theta_j) \text{ for } j = 1, \dots, N, \quad (1)$$

where θ_j represents the phase of the j th oscillator, ω_j represents the natural frequency of the j th oscillator, $K > 0$ represents the coupling strength, and h denotes the coupling function. In the Kuramoto model where $h(x) = \sin(x)$ [14, 15], the oscillators are synchronized when the coupling strength K is sufficiently large. Synchronization transitions have been extensively studied with regard to their analogy to the second-order phase transition [14, 15]. A measure of synchrony is given by the order parameter $R(t)$ as follows [14]:

$$R(t) \equiv \frac{1}{N} \left| \sum_{j=1}^N \exp(2\pi i \theta_j) \right|. \quad (2)$$

A synchronization transition is characterized by a change in the order parameter $R(t)$ from almost zero to a finite value with an increase in the coupling strength K . In the thermodynamic limit $N \rightarrow \infty$, the stationary state of $R(t)$ in the incoherent (desynchronized) regime bifurcates at the critical coupling strength $K = K_c$, above which the oscillators are coherent (synchronized). When N is finite, the effect of an increase in N on fluctuations can be different between the incoherent and coherent regimes [1, 12]. To illustrate this, we introduce the diffusion coefficient of the time integral of $R(t)$, characterizing long-term fluctuations of $R(t)$ in systems of finite size [12]. We consider the variance of the time integral $\int_0^t R(s)ds$, defined as follows:

$$\sigma^2(t) \equiv N \left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\int_{t_0}^{t+t_0} R(s)ds - \langle R \rangle_t t \right)^2 dt_0 \right], \quad (3)$$

where $\langle \cdot \rangle_t$ represents the long-time average, and $[\cdot]_s$ denotes the sample average over different realizations of ω_j that are chosen from a certain distribution. For the variance in Eq. (3), the following diffusion law holds [12]:

$$D \equiv \lim_{t \rightarrow \infty} \sigma^2(t)/2t, \quad (4)$$

which represents the deviation of $\int_0^t R(s)ds$ from its mean value $\langle R \rangle_t t$ on the sample average. The standard scaling law of D with respect to system size N is described as follows [12]:

$$D \sim O(1). \quad (5)$$

Namely, the value of D is almost constant independently of the system size N . We numerically confirmed that this scaling law is observed in the Ising model [1], the probabilistic cellular automaton model [1, 21, 22], and the random neural network model [23]. In the previous study [12], we also confirmed that the standard scaling law holds in the incoherent regime of the globally coupled phase oscillator model (1). In our discussion in Ref. [12], we suggested that

the standard scaling law can be observed in a wide range of systems. We also found that, in the coherent regime of system (1), the asymptotic form of D for large N is represented as

$$D \sim O(1/N^a), \quad (6)$$

where $a > 0$ is a certain constant. This non-standard scaling law suggests the existence of a new universality class. However, it is still obscure how broadly this scaling law is observed in coupled oscillator systems. The aim of this study is to elucidate the validity range of the above non-standard scaling law of D .

II. NECESSARY CONDITION FOR NON-STANDARD SCALING LAW

Let us consider the mean frequencies $\tilde{\omega}_j = \lim_{t \rightarrow \infty} (\theta_j(t) - \theta_j(0))/t$ of the phase variables in the stationary state. We define by $\Gamma(\tilde{\omega})$ the distribution function of the mean frequencies of all the oscillators. A necessary condition for the scaling law (6) in system (1) is given as follows [12]:

$$\lim_{\tilde{\omega} \rightarrow \Omega} \Gamma(\tilde{\omega}) = 0 \text{ with } N \rightarrow \infty, \quad (7)$$

where Ω denotes the common frequency of the entrained oscillators. A typical example is illustrated in Figs. 1(a) and 1(b), where we observe that Eq. (7) holds in a coherent state whereas it does not hold in an incoherent state [2]. In Fig. 1(a), oscillators with $\tilde{\omega} \sim 0$ are dominant and oscillators with fast oscillations very few. In Fig. 1(b), we can observe the largest peak corresponding to entrained oscillators and two other peaks corresponding to non-entrained oscillators with fast oscillations. It has been previously shown that the value of D is mostly determined by the value of $\Gamma(\tilde{\omega})$ around $\tilde{\omega} = \Omega$ [12]. The value of D usually decreases with system size N , if Eq. (7) holds.

To understand the difference between the standard and non-standard scaling properties, we consider the following correlation function of the order parameter:

$$C(t) \equiv N \left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (R(t+t_0) - \langle R \rangle_t)(R(t_0) - \langle R \rangle_t) dt_0 \right]_s. \quad (8)$$

Consequently, we can relate $C(t)$ to D as follows [12, 25]:

$$D = \frac{1}{2} \int_{-\infty}^{\infty} C(s) ds. \quad (9)$$

Therefore, the form of the correlation function $C(t)$ determines the value of D . In the incoherent regime of system (1), the form of $C(t)$ is almost exponential as shown in Fig. 1(c) [9, 12]. In contrast, in the coherent regime, the non-entrained oscillators generate fluctuations in the correlation function $C(t)$ around $C(t) = 0$ as shown in Fig. 1(d) [9, 12]. Therefore, a characteristic time period with $C(t) < 0$ is observed.

III. GENERALITY OF THE NON-STANDARD SCALING LAW

Let us discuss what type of coupled oscillator models satisfy the condition (7). It has been reported that this condition is not satisfied in a coupled oscillator model with local coupling [18], one with very non-uniform coupling strength [24], and one with a large variance in the natural frequencies [27]. To avoid these cases, it appears to be necessary that a model involves global coupling [2, 7, 14, 16, 30], nearly uniform coupling strength [31], and a sufficiently small variance in the natural frequencies [29].

We examine the existing models that are known to satisfy the condition (7) in order to confirm whether the scaling law (6) holds or not. The results of our numerical simulations, presented in the following part, suggest that the condition (7) is not only a necessary but also a sufficient condition for the non-standard scaling law.

First, we investigate the phase oscillator model with time-delayed coupling [28], described as follows:

$$\dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{k=1}^N \sin(\theta_k(t - \tau) - \theta_j(t)), \quad (10)$$

where τ represents the time delay. Equation (7) holds for any time delay τ [28]. Second, we consider the phase oscillator model with inertia [26] given by

$$m\ddot{\theta}_j + \dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{k=1}^N \sin(\theta_k - \theta_j), \quad (11)$$

where m corresponds to the strength of the inertia. The inertia introduces an adaptive effect in the system [26]. Numerical simulations have confirmed that Eq. (7) holds even for large values of m , i.e., for strong adaptive effects [26]. Third, we explore the effect of amplitudes by using the globally coupled Stuart–Landau equations [27], described as follows:

$$\dot{z}_j = (1 - |z_j|^2 + i\omega_j)z_j + \frac{K}{N} \sum_{k=1}^N (z_k - z_j), \quad (12)$$

where $z_j \in \mathcal{C}$ represents the complex state variable of the j th oscillator. The variance of the natural frequencies ω_j is assumed to be not so large; otherwise, collective chaos or oscillation death occurs and violates Eq. (7) [27].

We show that all of these models satisfy the scaling law (6). The value of D is estimated by fitting $\sigma^2(t)$ with a line of slope $2Dt$ except for the transient period, as shown in Fig. 2. For all the systems considered in the present paper, we clearly see that D is scaled as $D \sim O(1/N^a)$, as shown in Figs. 3(a)-(c). In addition, the correlation function has a characteristic time period in which $C(t) < 0$ as shown in Figs. 3(d)-(f). Therefore, it is suggested that Eq. (7) is a sufficient condition for $D \sim O(1/N^a)$ in globally coupled oscillator systems with large N .

IV. DISCUSSION

We have investigated the condition under which the non-standard scaling law (6) for the statistical quantity D holds. We have considered the relation between the non-standard scaling law and the condition in Eq. (7), and we have performed numerical simulations of several models satisfying the condition. The numerical results suggest that the condition is not only a necessary but also a sufficient condition for the non-standard scaling law. Our non-standard scaling law (6) does not sensitively depend on details of coupling schemes [12] as well as on other factors such as time delay, inertia, and amplitude variables as demonstrated in this study. It is noteworthy that such a non-standard scaling law that can be observed in a large class of globally coupled oscillator models has not been reported thus far. In fact, even for conventional statistical quantities such as the variance and the correlation time of the order parameter, scaling laws with respect to coupling strength K and those with respect to system size N have not been fully explored except for those of the phase oscillator model (1) [2, 9–12, 15, 17–20]. Thus, our result is useful for understanding fluctuations in general systems of globally coupled oscillators.

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- [1] H. Nishimori and G. Ortiz, *Elements of Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 2010).
 - [2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press, Cambridge, England, 2001).
 - [3] A. von Oertzen, H. H. Rotermund, A. S. Mikhailov, and G. Ertl, *J. Phys. Chem. B* **104**, 3155 (2000).
 - [4] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1981).
 - [5] J. Buck, *Q. Rev. Biol.*, **63**, 265 (1988).
 - [6] L. Glass, P. Hunter, and A. McCulloch, editors. *Theory of Hearts* (Springer, Verlag, 1990).

- [7] H. Daido, Phys. Rev. Lett. **73**, 760 (1994); Physica D **91**, 24 (1996).
- [8] J. D. Crawford, Phys. Rev. Lett. **74**, 4341 (1995); J. D. Crawford and K. T. R. Davies, Physica D **125**, 1 (1999).
- [9] H. Daido, J. Stat. Phys. **60**, 753 (1990).
- [10] E. J. Hildebrand, M. A. Buice, and C. C. Chow, Phys. Rev. Lett. **98**, 054101 (2007).
- [11] M. A. Buice and C. C. Chow, Phys. Rev. E **76**, 031118 (2007).
- [12] I. Nishikawa, G. Tanaka, T. Horita, and K. Aihara, Chaos **22**, 013133 (2012).
- [13] H. Chiba and I. Nishikawa, Chaos **21**, 043103 (2011).
- [14] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984).
- [15] S. H. Strogatz, Physica D **143**, 1 (2000); J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005).
- [16] H. Sakaguchi and Y. Kuramoto, Prog. Theor. Phys. **76**, 576 (1986).
- [17] A. Pikovsky and S. Ruffo, Phys. Rev. E **59**, 1633 (1999).
- [18] H. Hong, H. Chaté, H. Park, and L.-H. Tang, Phys. Rev. Lett. **99**, 184101 (2007).
- [19] S.-W. Son and H. Hong, Phys. Rev. E **81**, 061125 (2010).
- [20] I. Nishikawa, K. Iwayama, G. Tanaka, T. Horita, and K. Aihara, “Finite-size scaling in globally coupled phase oscillators with a general coupling scheme”, Arxiv preprint arXiv:1211.4364 (2012).
- [21] F. Bagnoli, N. Boccara, and R. Rechtman, Phys. Rev. E **63**, 046116 (2001).
- [22] K. Takeuchi, Physica D **223**, 146 (2006).
- [23] T. Toyozumi and L. F. Abbott, Phys. Rev. E **84**, 051908 (2011).
- [24] H. Daido, Phys. Rev. Lett. **68**, 1073 (1992).
- [25] R. Kubo, M. Toda, and N. Hashitsume, *Statistical physics II: Nonequilibrium statistical mechanics* (Springer, 1985).
- [26] H.-A. Tanaka, A. J. Lichtenberg, and S. Oishi, Phys. Rev. Lett. **78**, 2104 (1997).
- [27] P. C. Matthews, R. E. Mirollo, and S. H. Strogatz, Physica D **52**, 293 (1991).
- [28] M. Y. Choi, H. J. Kim, D. Kim and H. Hong, Phys. Rev. E **61**, 371 (2000).
- [29] K. H. Nagai, H. Kori, Phys. Rev. E **81**, 065202(R) (2010).
- [30] H. Daido, Prog. Theor. Phys. **88**, 1213 (1992).
- [31] T. Aonishi, K. Kurata, and M. Okada, Phys. Rev. Lett. **82**, 2800 (1999).

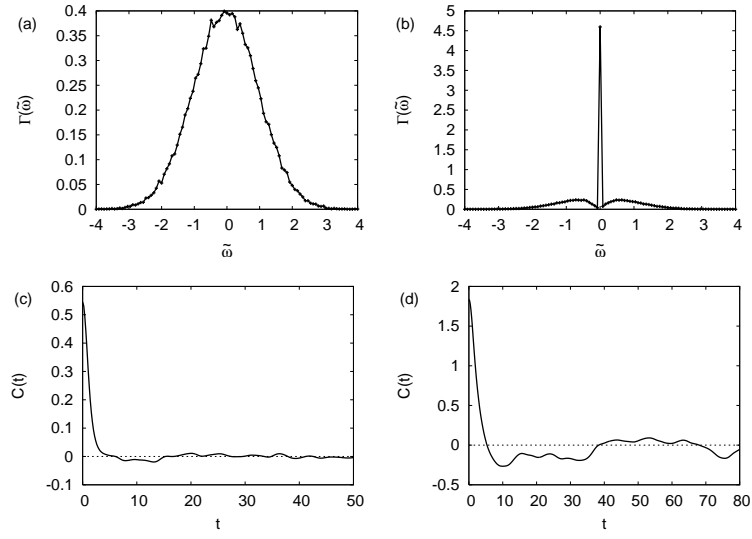


FIG. 1. Numerical results of $\Gamma(\tilde{\omega})$ (upper panels) and the time evolution of the correlation function $C(t)$ (lower panels) for the Kuramoto model (Eq. (1) with $h(x) = \sin(x)$) with $N = 64000$. The natural frequencies ω_j are randomly drawn from a Gaussian distribution with mean zero and standard deviation one. (a) and (c) An incoherent state with $K = 0.8$. (b) and (d) A coherent state with $K = 1.63$

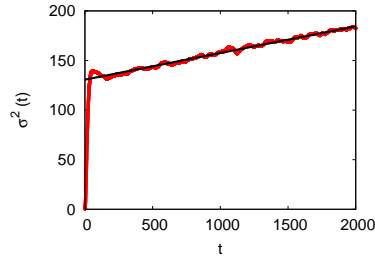


FIG. 2. The time evolution of the variance $\sigma^2(t)$ of the integrated order parameter in the coherent regime of system (12) with amplitude variables, where $N = 8000$ and $K = 0.31$. The value of $\sigma^2(t)$ increases linearly with slope $2Dt$ after a transient period. The natural frequencies ω_j are randomly generated from a Gaussian distribution $G(\omega)$ with mean 0 and standard deviation 0.2. The red line shows an average over 300 different initial conditions.

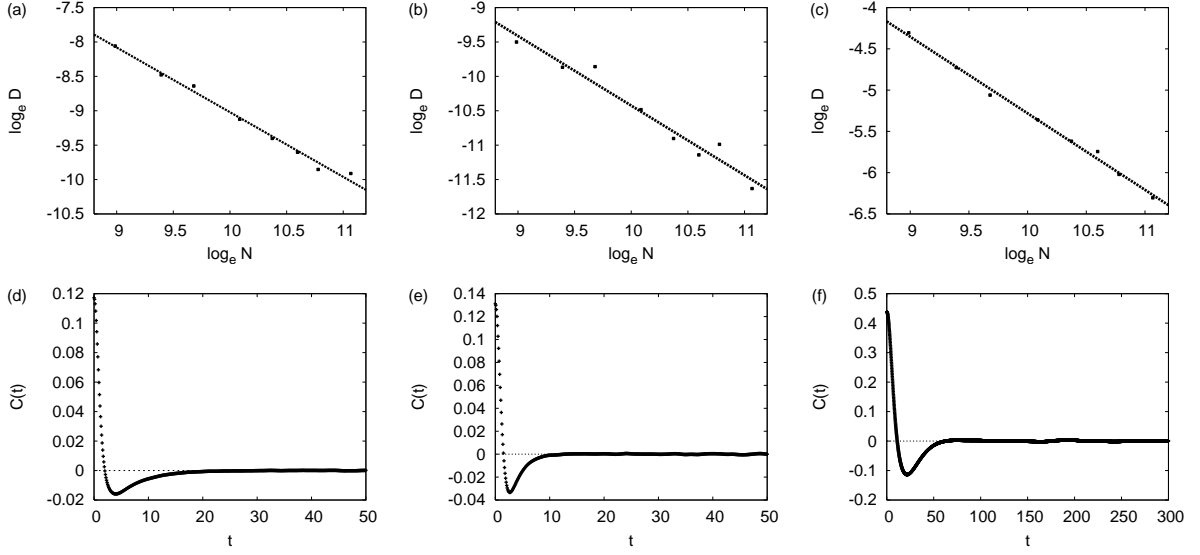


FIG. 3. Behavior of statistical quantities in the coherent regime. The upper panels show the scaling properties of D with respect to the system size N ranging from 8000 to 64000. The lower panels show the correlation function $C(t)$ for $N = 64000$. Each plot is an average over 100 different initial conditions. The diffusion coefficient D is scaled as $D \sim O(1/N^a)$ with $a \sim 0.940$ for (a), $a \sim 1.012$ for (b), and $a \sim 0.927$ for (c). (a) and (d) System (10) with time delayed coupling, where time delay $\tau = 1$ and $K = 1.78$. We used the order parameter given by Eq. (2). (b) and (e) System (11) with inertia term, where $m = 0.1$ and $K = 1.93$. We used the order parameter given by Eq. (2). (c) and (f) System (12) with amplitude variables, where $K = 0.325$. We used the order parameter defined as $R(t) \equiv \left| \sum_{j=1}^N z_j \right| / N$. The natural frequencies ω_j are randomly generated from a Gaussian distribution $G(\omega)$ with mean zero and a fixed standard deviation. The standard deviation of the natural frequencies ω_j is set at 1 for systems (10) and (11), and at 0.2 for system (12). In the case where the natural frequencies are given deterministically i.e. $j/(N+1) = \int_{-\infty}^{\omega_j} G(\tilde{\omega}) d\tilde{\omega}$, the result does not change.